

CANONICAL INFINITESIMAL DEFORMATIONS

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CONTENTS

0. Introduction	1
1. Coalgebra	6
2. Products	9
2.1. Very symmetric products	9
2.2. Jacobi complex	11
2.3. Obstructions	13
3. Second order	14
4. n -th order	18
5. Generalizations	25
References	27

0. INTRODUCTION

Our purpose here is to develop a 'canonical' approach to infinitesimal and formal deformation theory. For simplicity we shall stick in the paper mainly to one fundamental-and somewhat typical-case, that of a compact complex manifold X without global vector fields. Our starting point, and model, is the classical (first-order) Kodaira-Spencer formalism: this associates to any deformation \mathfrak{X}/S with

special fiber X a 'Kodaira-Spencer' map

$$\kappa : T_0 S \rightarrow H^1(X, T)$$

where $T = T_X$ is the tangent sheaf, and consequently obtains a canonical identification between the set of first-order deformations of X and the cohomology group $H^1(X, T)$. It is then natural to seek a higher-order analogue of this, for n -th order tangent spaces and n -th order deformations. At a minimum, one would like an n -th order analogue of κ :

$$\kappa_n = \kappa_n(\mathfrak{X}/S) : T^{(n)}S \rightarrow (?)_n$$

where $(?)_n$ is an explicit (and preferably computable) cohomological functor of X which, at least in favorable cases (e.g. when a global moduli space exists), should be canonically identifiable with the n -th order tangent space at a smooth point of the moduli space. Put another way, one knows, when $H^0(T) = 0$, the *existence* of an universal formal deformation

$$\hat{X}/\hat{S} = \lim_n X_n/S_n.$$

The problem is to write each X_n/S_n , i.e, the universal n -th order deformation, as an explicit cohomological functor of X , extending the above Kodaira-Spencer identification of first-order deformations.

Our approach to this is to combine some earlier constructions from $[R_1]$ with an important and very novel insight coming out of recent work of Beilinson, Drinfeld and Ginzburg, cf. $[BG]$. The latter is, among other things, concerned specifically with deformations of vector bundles and principal bundles on a fixed complex curve X . It gives a formula for the n -th order cotangent space to the moduli of such bundles as H^0 of a suitable sheaf on the Knudsen-Mumford space \hat{X}^n parametrizing

n -tuples of points on X . This suggests the simple but stunning—to the author—and very broad philosophy that n -th order deformations should be related to n -tuples: e.g. that $T^{(n)}M$ or something similar ought to be writable in terms of cohomology, at least on some sort of parameter space for n -tuples on X (notwithstanding that a good analogue of \hat{X}^n is not known if $\dim X > 1$).

Here we will realize this philosophy as follows. First, we construct certain spaces $X < n >$, related to symmetric products, which we call the *very symmetric products* of X . To be precise, $X < n >$ parametrizes the nonempty subsets of X of cardinality $\leq n$. These naturally form a tower:

$$X = X < 1 > \subset X_2 = X < 2 > \subset X < 3 > \cdots \subset X < n > \cdots \subset X < \infty > =$$

$$\varinjlim X < n > .$$

Then on $X < \infty >$ we construct a certain complex $J = J^*(T_X)$ which we call the *Jacobi complex* of X : this is essentially just a multivariate version of the standard complex used to compute the *Lie algebra homology* of T_X . cf. [F] (indeed the latter homology coincides with the cohomology of J^* along $X < 1 >$). The subcomplex $F^n J^* =: J_n^*$ is naturally supported on $X < n > \subset X < \infty >$. With this, we will prove the following

Theorem 0.1. *Let X be a compact complex manifold with $H^0(T_X) = 0$ and let J be the Jacobi complex of X . Then*

(i) *for each n there is a canonical ring structure on*

$$R_n^u = \mathbb{C} \oplus H^0(J_n^*)^*$$

and a canonical flat deformation X_n^u/R_n^u , these fit together to form a direct system with limit

$$\hat{X}^u/\hat{R}^u = \lim_n X_n^u/R_n^u ;$$

(ii) for any artin local \mathbb{C} -algebra R_n of exponent n and flat deformation X_n/R_n of X , there is a canonical Kodaira-Spencer ring homomorphism

$$\alpha_n = \alpha_n(X_n/R_n) : R_n^u \rightarrow R_n$$

and an isomorphism

$$X_n/R_n \xrightarrow{\sim} \alpha_n^* X_n^u = X_n^u \times_{R_n^u} R_n ;$$

(iii) if $\hat{R} = \varprojlim R_n$ is a complete local noetherian \mathbb{C} -algebra and $\hat{X} = \lim X_n/R_n$, then $\hat{\alpha} = \lim_n \alpha_n : \hat{R}^u \rightarrow \hat{R}$ exists and $\hat{X}/\hat{R} = \hat{\alpha}^*(\hat{X}^u/\hat{R}^u)$

The result naturally generalizes (cf. Sect. 5). If g is a sheaf of \mathbb{C} -Lie algebras on X and E a g -module (both assumed reasonably 'tame'), let \bar{g} be the unique quotient of g acting faithfully on E and assume that $H^0(\bar{g}) = 0$. For any artin local \mathbb{C} -algebra (R, m) of exponent n we may define a sheaf of groups G_R on X by

$$G_R = \exp(g \otimes m)$$

(i.e. G_R is $g \otimes m$ with multiplication determined by the Campbell-Hausdorff formula) and similarly $\bar{G}_R = \exp(\bar{g} \otimes m)$. Then g -deformations of E over R are locally trivial deformations with transitions in \bar{G}_R , and are naturally classified by the nonabelian Čech cohomology set $H^1(X, \bar{G}_R)$. Our construction yields a bijection $v = v_{R,E}$ between these and a certain subset of $\mathbb{H}^0(J_n(\bar{g})) \otimes m$ (i.e. the set of 'morphic' elements). For $n \geq 3$ this correspondence is given somewhat indirectly and in particular does not come from an explicit correspondence on the cocycle

level. v apparently depends on both E and g , though its source and target depend only on \bar{g} . It is unknown to the author whether (say for E a faithful g -module) v is independent of E , a fortiori whether it can be defined in terms of g alone. For another, perhaps more 'conceptual' interpretation of our construction of the universal deformation of E , see [R3], Theorem 3.1.

A further generalization, to the case of a sheaf of differential graded Lie algebras, will be considered in [R3].

As indicated above, the existence of the universal formal deformation \hat{X}^u/\hat{R}^u was known before, thanks to the work of Grothendieck, Schlessinger et al.: our point is its explicit construction and description. As for applications and extensions of the method, these have been, and will be given elsewhere, but a few can be mentioned here.

(i) An analogous deformation theory for deformations of vector bundles (or more generally locally free sheaves over a fixed locally \mathbb{C} -ringed space) and, as one application, construction of a symplectic (closed) 2-form on the moduli space, generalising at the same time constructions of Hitchin (for local systems on Riemann surfaces) and Mukai (for holomorphic vector bundles over K3 surfaces)[R5].

(ii) A direct construction of the universal variation of Hodge structure associated to a compact Kahler manifold and resulting study of the (local) period map and characterisation of its image (local Schottky relations), especially for Calabi-Yau manifolds and curves [R3],[L].

(iii) A theory of semiregularity for submanifolds and embedded, as well as relative deformations and resulting dimension bounds for Hilbert schemes and relative deformation spaces [R2][R4].

Higher-order Kodaira-Spencer maps, especially associated to 'geometric' (reduced) families have been independently defined by Esnault and Viehweg, cf. [EV]; that paper defines higher-order (additive) Kodaira-Spencer classes, but does not construct the universal family. Some important antecedents (albeit from a different viewpoint) are in the work of Goldman-Millson [GM].

1. COALGEBRA

The purpose of this section is to characterize the vector space m^* dual to the maximal ideal of an artin local \mathbb{C} -algebra (R, m) . While the general concept of coalgebra is well known, our application in the artin local case assigns a special role to the m -adic filtration and its dual, the 'order' filtration, not present in the general case. Consequently, it will be convenient to give a brief self-contained treatment here.

By an Order-Symbolic (OS) structure of order n we mean a finite-dimensional \mathbb{C} -vector space together with an increasing filtration

$$V^0 = 0 \subseteq V^1 \subseteq \dots \subseteq V^n = V$$

and mutually compatible 'symbol' or 'comultiplication' maps

$$\sigma^{i,j} : V^i/V^j \rightarrow S^2(V^{i-j}), \quad j < i.$$

(Sometimes we shall use the same notation to denote the induced map $V^i \rightarrow S^2(V^i)$; actually, a moment's thought shows that $\sigma := \sigma^{n,1}$ is sufficient to determine the rest). These are assumed to satisfy the natural (co)associativity condition that the

following diagram should commute

$$\begin{array}{ccc}
 & S^2(V^{n-1}) \otimes V^{n-1} & \\
 \nearrow \phi & & \searrow \\
 V/V^1 \rightarrow S^2 V^{n-1} \subset V^{n-1} \otimes V^{n-1} & & V^{n-1} \otimes V^{n-1} \otimes V^{n-1} \\
 \searrow \psi & & \nearrow \\
 & V^{n-1} \otimes S^2(V^{n-1}) &
 \end{array}$$

$$\varphi = \sigma^{n-1,1} \otimes id, \quad \psi = id \otimes \sigma^{n-1,1}$$

An OS structure V is said to be *standard* if σ is injective (hence $\sigma^{n,j}$ is injective for all $j < n$). We can now state the basic result about OS structures, which relates them with artin local algebras.

Proposition 1.1. *There is an equivalence of categories between \mathcal{OS}_n , the category of OS structures of order n , and \mathcal{FR}_n , the category of commutative artin local \mathbb{C} -algebras space (R, m) of exponent n together with a super- m -adic filtration $(m_i \supseteq (m)^i)$, where standard structures correspond with m -adically filtered algebras. The correspondence is given by*

$$(V, V^*, \varphi) \mapsto (\mathbb{C} \oplus V^*, V^{i\perp} = (V/V^i)^*, \varphi_n^*)$$

$$(S, m, m_\bullet) \mapsto (m^*, m_{i-1}^\perp = (m/m_{i-1})^*, \text{comultiplication}).$$

Proof. Basically trivial. Given V etc. define

$$R = \mathbb{C} \oplus V^*, m = V^*, m_i = (V^{i-1})^\perp = (V/V^{i-1})^* \subset V^*$$

Dualising σ yields the multiplication map

$$S^2(m) \rightarrow S^2(m/m_n) \xrightarrow{\sigma^*} m_2 \subset m$$

This extends in an obvious way to a commutative associative multiplication map $S^2 R \rightarrow R$. By construction, σ^* descends to a map

$$S^2(m/m_i) \xrightarrow{\sigma^{i+1,2*}} m_2/m_{i+1}$$

hence $m \cdot m_i \subset m_{i+1}$. So inductively m_i is firstly an ideal and then $m_i \supseteq m^i$ by induction. The rest is similar. \square

Thus in particular, to an artin local \mathbb{C} -algebra (R, m) of exponent n , we have a uniquely determined standard OS structure on $T^n R = m^*$, which conversely determines (R, m) . For later use it is convenient to explicate and amplify the morphism part of the above equivalence.

Corollary 1.2. *Let $(R, m), (R', m')$ be artin local algebras of exponent n . Then the following are mutually interchangeable:*

- (i) *a local homomorphism $\eta : R' \rightarrow R$;*
- (ii) *an OS morphism $\kappa : T^n R \rightarrow T^n R'$;*
- (iii) *a compatible collection of elements*

$$v_i \in m^{n+1-i} \otimes T^n R' / T^{n-i} R'$$

$$\text{such that } (id \otimes \sigma)(v_n) = v_n \cdot v_n \in m^2 \otimes S^2(T^n R') \quad (1.1)$$

Proof. Only (iii) may require comment. v_n evidently determines κ as well as v_1, \dots, v_{n-1} ; it is the existence of the latter that ensures that κ is filtration-preserving, while (1.1) makes κ compatible with comultiplication. \square

Let us call an element $v \in m_R \otimes T^n R'$ as above *morphic*.

2. PRODUCTS

2.1. Very symmetric products. Fix a topological space X . For any $n \geq 1$, we denote by X^n and X_n the Cartesian and symmetric products, respectively. The system $(X^n, n \in \mathbb{N})$ forms essentially a *simplicial* configuration (while the X_n 's are related to one another in even more complicated ways). On the other hand, the system of the n -th order neighborhoods of a point (say on a moduli space), $n \in \mathbb{N}$, is simply a *tower*. This indicates that the 'right' spaces of point-configurations to work with in deformation theory are neither X^n or X_n but a suitable modifications thereof which form a tower. We now proceed to define these spaces which we call the *very symmetric products* (powers) of X and denote by $X < n >$. A word to the wise: defining $X < n >$ may appear to be a fastidious bother as (sheaf) cohomology behaves simply with respect to finite maps; however, it is *complexes* that we must work with, and even to define the coboundary maps in appropriate complexes, a certain minimum amount of 'quotienting' must be effected , e.g. it seems that the Jacobi complexes defined below on $X < n >$ cannot be defined on any natural space strictly 'above' $X < n >$ (and this certainly includes X_n).

As a set , we define

$$X < n > = X^n / \sim$$

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \text{ iff } \{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}.$$

Thus $X < n >$ parametrises precisely the nonempty subsets of X of cardinality $\leq n$. We endow $X < n >$ with the quotient topology induced from X^n . Note that we indeed have a tower of (closed, for X separated) embeddings

$$X = X < 1 > \subset X_2 = X < 2 > \subset X < 3 > \cdots \subset X < n > \cdots \subset X < \infty > =$$

$$\lim_{\rightarrow} X < n > .$$

Alternatively, $X < n >$ may be defined inductively: let

$$X^{n-1} \rightarrow X^n \rightarrow X_n$$

be a 'diagonal' map, e.g. $(x_1, \dots, x_{n-1}, x_{n-1}) \mapsto \{x_1, \dots, x_{n-1}\}$, whose image D_{n-1} is the big 'diagonal' in X_n (and is independent of the choice of which point gets doubled). Then

$$\begin{aligned} X < n > &= X_n \bigcup_{X^{n-1}} X < n-1 > \\ &= X_n \bigcup_{D^{n-1}} X < n-1 > \end{aligned} \quad (2.1)$$

(it is easy to see inductively that the natural map $q_n : X_n \rightarrow X < n >$ factors through D_n). Via (2.1), very symmetric products may be defined in more general settings, e.g. when X is a Grothendieck topology.

It is not hard to see that if X has a structure of (separated) analytic space, then so does $X < n >$ in a natural way. However, we shall not need this fact. Rather, the sheaves on $X < n >$ relevant to us will be alternating products of sheaves induced from X , which now proceed to define. Let S be a ring and A a sheaf of S -modules on X . Let $\pi_n : X^n \rightarrow X < n >$ be the natural map, and set

$$\tau_S^n(A) = \pi_{n*}(A \boxtimes_S \cdots \boxtimes_S A)$$

(When S is understood, e.g. $S = \mathbb{C}$, we may suppress it). Note that the symmetric group Σ_n acts in a natural way on $\tau_S^n(A)$ and let $\sigma_S^n(A)$ (resp. $\lambda_S^n(A)$) denote the invariant and antiinvariant factors. Note that this definition makes sense on the symmetric product X_n already, and may also be extended to mixed (Schur) tensors in an obvious way.

When A is replaced by a complex A^\cdot of S -modules, these constructions extend in a natural way to make $\tau_S^n(A^\cdot), \sigma_S^n(A^\cdot), \lambda_S^n(A^\cdot)$ into complexes; for instance

$$\lambda_S^2(A^\cdot) = \lambda_S^2(A^{\text{even}}) \oplus \pi_{2*}(A^{\text{even}} \boxtimes A^{\text{odd}}) \oplus \sigma_S^2(A^{\text{odd}}).$$

The cohomology of $\tau_S^n(A)$ can be computed by the Künneth formula, at least if A is S -free, i.e.

$$H^m(\tau_S^n(A)) = [\otimes_1^n H^\cdot(A)]^m$$

In fact the n -th tensor power of a Čech complex for A (with respect to an acyclic cover of X) yields one for $\tau_S^n(A)$. As everything decompose into \pm eigenspaces under the action of Σ_n , analogous comments apply to $\sigma_S^n(A)$ and $\lambda_S^n(A)$ (one must take into account the usual sign rules for cup products, e.g. $a \cup b = (-1)^{\deg a \deg b} b \cup a$). For instance, in the case of principal interest to us, we have $H^0(A) = 0$ and then

$$H^i(X < n >, \lambda_S^n(A)) = H^i(X < n >, \tau_S^n(A)) = 0, i < n;$$

$$H^n(X < n >, \lambda_S^n(A)) = S_S^n H^1(A) :$$

in fact, the symmetric power of the Čech complex for A may be used to compute the cohomology of $\lambda_S^n(A)$.

Remark The spaces $X < n >$ have recently appeared in the work of Beilinson and Drinfeld on 'Chiral Algebras'; I am grateful to V. Ginzburg for pointing this out.

2.2. Jacobi complex. Let \mathcal{L}^\cdot be a sheaf of complex differential graded Lie algebras (DGLAs) on X . Thus \mathcal{L}^\cdot is a "lie object" in the category of complexes of \mathbb{C} -modules on X , which means there is a morphism $bt : \Lambda^2(\mathcal{L}^\cdot) \rightarrow \mathcal{L}^\cdot$, whose natural extension as a derivation of degree -1 on the Grassmann algebra $\oplus \Lambda^i(\mathcal{L}^\cdot)$ satisfies $bt^2 = 0$;

or course 'Grassmann algebra' and wedge must be understood in the graded sense, compatible with the gradation on \mathcal{L}^\bullet . Note that br induces a map

$$br : \lambda^2(\mathcal{L}^\bullet) \rightarrow \Lambda^2(\mathcal{L}^\bullet) \rightarrow \mathcal{L}^\bullet$$

(i.e. restriction followed by br). Now we associate to \mathcal{L}^\bullet a complex $J(\mathcal{L}^\bullet)$ on $X < \infty >$ called the Jacobi complex of \mathcal{L}^\bullet , as follows. Set

$$J^{-n}(\mathcal{L}) = \lambda^n(\mathcal{L}^\bullet), \quad n \geq 1$$

(where the latter is viewed as a sheaf on $X < \infty >$ via $X < n > \subset X < \infty >$ and λ is understood in the graded sense); the differential $d_n : \lambda^n(\mathcal{L}^\bullet) \rightarrow \lambda^{(n-1)}(\mathcal{L}^\bullet)$ is defined as follows. First, let $alt : \tau^2(\mathcal{L}^\bullet) \rightarrow \lambda^2(\mathcal{L}^\bullet)$ be the alternation or skew-symmetrization map, where $\lambda^2(\mathcal{L}^\bullet)$ is viewed as a complex on $X < 2 >$ via the diagonal embedding $X \rightarrow X < 2 >$, and set

$$a = \pi_{n*}(id \boxtimes alt) : \pi_{n*}(\boxtimes^n \mathcal{L}^\bullet) \rightarrow \pi_{n*}(\boxtimes^{n-2} \mathcal{L}^\bullet \boxtimes \lambda^2(\mathcal{L}^\bullet)).$$

Next, note that $\pi_{n*}(\boxtimes^{n-2} \mathcal{L}^\bullet \boxtimes \lambda^2(\mathcal{L}^\bullet))$ as defined above coincides with $\pi_{(n-1)*}(\boxtimes^{n-2} \mathcal{L}^\bullet \boxtimes \lambda^2(\mathcal{L}^\bullet))$ and set

$$b = \pi_{(n-1)*}(id \boxtimes br) : \pi_{(n-1)*}(\boxtimes^{n-2} \mathcal{L}^\bullet \boxtimes \lambda^2(\mathcal{L}^\bullet)) \rightarrow \pi_{(n-1)*}(\boxtimes^{n-2} \mathcal{L}^\bullet \boxtimes \mathcal{L}^\bullet) = \tau^{n-1}(\mathcal{L}^\bullet).$$

Finally let $p : \tau^{n-1}(\mathcal{L}^\bullet) \rightarrow \lambda^{n-1}(\mathcal{L}^\bullet)$ be the natural alternation map and $i : \lambda^n(\mathcal{L}^\bullet) \rightarrow \tau^n(\mathcal{L}^\bullet)$ the inclusion. Then define

$$d_n = p \circ b \circ a \circ i.$$

More explicitly,

$$d_n(t_1 \times \cdots \times t_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) [t_{\sigma(1)}, t_{\sigma(2)}] \times t_{\sigma(3)} \times \cdots \times t_{\sigma(n)}$$

The Jacobi identity for \mathcal{L}^\bullet ensures that $J(\mathcal{L}^\bullet)$ is a complex. Put $J_n(\mathcal{L}^\bullet) = J^{\geq -n}(\mathcal{L}^\bullet)$, which may be viewed as complex on $X < n >$.

Now note that, viewing $J_n(\mathcal{L}^\bullet)/J_1(\mathcal{L}^\bullet) = J^{-n \leq \cdot \leq -2}(\mathcal{L}^\bullet)$ as a complex on $X < 2n - 2 >$, it forms a subcomplex of $\pi_{n-1, n-1*}(\text{Sym}^2(J_{n-1}(\mathcal{L}^\bullet)))$, where $\pi_{n-1, n-1} : X < n - 1 > \rightarrow X < 2n - 2 >$ is the natural map. This gives rise, e.g. to a map

$$\sigma^n : \mathbb{H}^0(J_n(\mathcal{L}^\bullet))/\mathbb{H}^0(J_1(\mathcal{L}^\bullet)) \rightarrow \mathbb{H}^0((J_n/J_1)(\mathcal{L}^\bullet)) \rightarrow S^2\mathbb{H}^0(J_{n-1}(\mathcal{L}^\bullet))$$

which we call the symbol map associated to \mathcal{L}^\bullet , it is not hard to see that with this $V^n(\mathcal{L}^\bullet) = \mathbb{H}^0(J_n(\mathcal{L}^\bullet))$ forms an OS structure, which is standard provided $\mathbb{H}^{\leq 0}(\mathcal{L}^\bullet) = 0$. By Section 1 then we obtain an inverse system of artin local algebras

$$R_n(\mathcal{L}^\bullet) = \mathbb{C} \oplus V^n(\mathcal{L}^\bullet)^*$$

and their limit $\hat{R}(\mathcal{L}^\bullet)$ which might be called the deformation ring associated to \mathcal{L}^\bullet .

In particular, if X is a compact complex manifold, its tangent sheaf $T = T_X$ forms a Lie algebra under Lie bracket of vector fields, and we denote the associated Jacobi complexes by $J_{n,X}$ or J_n , the corresponding OS structure by V_X^n or V^n , and the corresponding ring by $R_{n,X}^u$ or R_n^u . As we shall see, when $H^0(T) = 0$ the latter turns out to be the base ring of the n -universal deformation of X .

2.3. Obstructions. Assume $H^0(\mathcal{L}) = 0$. Note that the long cohomology sequence associated to

$$0 \rightarrow J_{n-1}(\mathcal{L}) \rightarrow J_n(\mathcal{L}) \rightarrow \lambda^n(\mathcal{L})[n] \rightarrow 0, n \geq 2,$$

gives rise to a 'big obstruction' map

$$Ob_n : \text{Sym}^n H^1(\mathcal{L}) \rightarrow \mathbb{H}^1(J_{n-1}(\mathcal{L})).$$

Let $K^n = \ker(Ob_n)$, so that we have an exact sequence

$$0 \rightarrow V^{n-1} \rightarrow V^n \rightarrow K^n \rightarrow 0.$$

Then, using the 'comultiplicative' structure on $J_n(\mathcal{L})$ as above it is easy to see that

Ob_n factors through a map, denoted ob_n , called the 'small obstruction map

$$ob_n : K_{n-1}.H^1(\mathcal{L}) \rightarrow H^2(\mathcal{L}),$$

where $K^{n-1}.H^1(\mathcal{L})$ denotes the intersection of $Sym^n H^1(\mathcal{L})$ and $K^{n-1} \otimes H^1(\mathcal{L})$ considered as subspaces of $\otimes^n H^1(\mathcal{L})$, and we have $K^n = \ker ob_n$ as well. Thus K^n may be described inductively, starting with $K^1 = H^1(\mathcal{L})$, $ob_1 = 0$.

Similar comments apply if \mathcal{L} is replaced by a dgla \mathcal{L}^\cdot with $\mathbb{H}^{\leq 0}(\mathcal{L}^\cdot) = 0$.

3. SECOND ORDER

For $n = 1$, Theorem 0.1 reduces to standard first -order Kodaira-Spencer deformation theory. Before taking up the general n-th order case in the next section, we consider here the second-order case, which is relatively simple but already illustrates some of the ideas. Thus let us fix an artin local \mathbb{C} -algebra (R_2, m_2) with reduction $(R_1, m_1) = (R_2/m_2^2, m_2/m_2^2)$ as well as an acyclic (say polydisc) open cover (U_α) of X , to be used in computing Čech cohomology. To a flat deformation

$$X_2/R_2 = \text{Spec}(\mathcal{O}_2)$$

we seek to associate a Kodaira-Spencer homomorphism

$$\alpha_2 = \alpha_2(X_2/R_2) : R_2^u \rightarrow R_2$$

or equivalently (cf. Section 2) a morphic element

$$v_2 = v_2(X_2/R_2) \in m_2 \otimes H^0(J_2)$$

which is to be described by a hypercocycle

$$v_2 = (u, \frac{1}{2}u^2) \in \check{C}^1(T) \otimes m_2 \oplus S^2\check{C}^1(T) \otimes m_2^2 \subset \check{C}^0(J_2) \otimes m_2 \quad (3.1)$$

where $u = (u_{\alpha\beta})$ is required to be a lifting of

$$v_1 = (v_{1\alpha\beta}) \in \check{Z}^1(T) \otimes m_1,$$

a cocycle representing $\mathcal{O}_1 = \mathcal{O}_2 \otimes_{R_2} R_1$ (where \mathcal{O}_2 is the structure sheaf of X_2),

and u^2 means exterior cup product in the cochain sense, i.e

$$(u^2)_{\alpha\beta\gamma} = u_{\alpha\beta} \times u_{\beta\gamma} \in S^2\check{C}^1(T) \otimes m_2^2 \subset \check{C}^2(\lambda^2 T) \otimes m_2^2$$

Note that the particular form of (3.1) makes the morphicity of v_2 automatic provided it is a hypercocycle, which means explicitly

$$-\frac{1}{2}[u_{\alpha\beta}, u_{\beta\gamma}] = u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = \delta(u) \quad (3.2)$$

$$\delta = \check{C}\text{ech coboundary}$$

Note that the LHS of (3.2) depends only on the reduction $v_{1\alpha\beta}$ of $u_{\alpha\beta}$ mod m_2^2

Now to define (u) we proceed as follows. As \mathcal{O}_2/R_2 in a flat deformation of \mathcal{O} , it is locally trivial hence we have isomorphisms of R_2 -algebras

$$\psi_\alpha : \mathcal{O}_2(U_\alpha) \rightarrow \mathcal{O}(U_\alpha) \otimes_{\mathbb{C}} R_2$$

which give rise to a gluing cocycle given by

$$D_{\alpha\beta}^2 = \psi_\alpha \circ \psi_\beta^{-1} \in \text{Aut}_{R_2}(\mathcal{O}(U_\alpha \cap U_\beta) \otimes R_2)$$

which reduces mod m_2^2 to

$$D_{\alpha\beta}^1 = I + v_{1\alpha\beta} \in \text{Aut}_{R_1}(\mathcal{O}(U_\alpha \cap U_\beta) \otimes R_1),$$

a gluing cocycle defining \mathcal{O}_1 .

Now it is easy to see that $D_{\alpha\beta}^2$ is uniquely expressible in the form

$$\begin{aligned} D_{\alpha\beta}^2 &= \exp(u_{\alpha\beta}) \\ &= I + u_{\alpha\beta} + \frac{1}{2}u_{\alpha\beta}^2, \quad u_{\alpha\beta} \in m_2 \otimes T(U_\alpha \cap U_\beta) : \end{aligned} \quad (3.3)$$

indeed starting with an arbitrary lift $u'_{\alpha\beta}$ of $v_{1\alpha\beta}$ to $m_2 \otimes T(U_\alpha \cap U_\beta)$, $\exp(u'_{\alpha\beta})$ and $D_{\alpha\beta}^2$ are R_1 -algebra homomorphisms which agree mod m_2^2 , hence differ by an m_2^2 -valued derivation $t_{\alpha\beta}$ and we may set $u_{\alpha\beta} = u'_{\alpha\beta} + t_{\alpha\beta}$. Now we simply plug (3.3) into the cocycle equation for D^2 :

$$D_{\alpha\beta}^2 D_{\beta\gamma}^2 = D_{\alpha\gamma}^2 \quad (3.4)$$

which becomes,

$$\begin{aligned} I + u_{\alpha\beta} + u_{\beta\gamma} + \frac{1}{2}u_{\alpha\beta}^2 + u_{\alpha\beta}u_{\beta\gamma} + \frac{1}{2}u_{\beta\gamma}^2 &= I + u_{\alpha\gamma} + \frac{1}{2}(u_{\alpha\gamma})^2 \\ &= I + u_{\alpha\gamma} + \frac{1}{2}(u_{\alpha\beta}^2 + u_{\alpha\beta}u_{\beta\gamma} + u_{\beta\gamma}u_{\alpha\beta} + u_{\beta\gamma}^2) \end{aligned}$$

as $(u_{\alpha\beta})$ is a cocycle mod m_2^2 . This is obviously equivalent to (3.2). Thus v_2 is a hypercocycle, as claimed.

Now the foregoing argument can essentially be read backwards given a morphic element

$$v_2 \in m_2 \otimes H^0(J_2),$$

choose a representative for v_2 of the form

$$((u_{\alpha\beta}), (u'_{\alpha\beta\gamma})) \in \check{C}'(T) \otimes m_2 \oplus S^2\check{C}'(T) \otimes m_2^2 \subset \check{Z}^0(J_2),$$

where $(u_{\alpha\beta})$ is a lifting of $(v_{1\alpha\beta})$; thus compatibility with comultiplication yields that v_2 may also be represented by

$$((u_{\alpha\beta}), \frac{1}{2}(u_{\alpha\beta})^2).$$

Then simply setting $D_{\alpha\beta}^2 = \exp(u_{\alpha\beta})$, the cocycle condition (3.4) follows from the hypercocycle condition (3.2), so that $(D_{\alpha\beta}^2)$ yields a locally trivial flat deformation $X_2/R_2 = \text{Spec}\mathcal{O}_2$, which we denote by $\Phi_2(\alpha_2)$ (though it is yet to be established that this is independent of choices).

This construction applies in particular to the identity map $R_2^u \rightarrow R_2^u$, thus yielding a flat deformation over R_2^u which we call an *universal second order deformation* and denote by $X_2^u = \text{Spec}(\mathcal{O}_2^u)$. It is moreover clear by construction that $\Phi_2(\alpha_2) = \alpha_2^*(X_2^u/R_2^u)$ for any $\alpha_2 : R_2^u \rightarrow R_2$ and also that for any second-order deformation X_2/R_2 ,

$$X_2/R_2 \approx \alpha_2(X_2/R_2)^*(X_2^u/R_2^u) \approx \Phi_2(\alpha_2(X_2/R_2)).$$

Similarly,

$$\alpha_2(\Phi_2(\beta)) = \beta.$$

Thus α_2 and Φ_2 establish mutually inverse correspondences, albeit on the cocycle level. What has to be established is that this correspondence descends to cohomology, i.e. non-abelian cohomology of Aut-cocycles and hypercohomology respectively. In one direction, consider two cohomologous Aut-cocycles

$$D_{\alpha\beta}^2 \sim D_{\alpha\beta}^{2'} = A_\beta D_{\alpha\beta}^2 A_\alpha^{-1}$$

$A_\alpha \in \text{Aut}_{R_2}(\mathcal{O}(U_\alpha) \otimes R_2)$, as above uniquely expressible in the form $\exp(w_\alpha)$, $w_\alpha \in m_2 \otimes T(U_\alpha)$. Thus

$$\begin{aligned}
D_{\alpha\beta}^{2'} &= (I + w_\beta + \frac{1}{2}w_\beta^2)(I + u_{\alpha\beta} + \frac{1}{2}u_{\alpha\beta}^2)(I - w_\alpha + \frac{1}{2}w_\alpha^2) \\
&= I + (u_{\alpha\beta} + w_\beta - w_\alpha + \frac{1}{2}[w_\beta - w_\alpha, u_{\alpha\beta}] + \frac{1}{2}[w_\alpha, w_\beta]) + \frac{1}{2}(u_{\alpha\beta} + w_\beta - w_\alpha)^2 \\
&= \exp(u_{\alpha\beta} + w_\beta - w_\alpha + \frac{1}{2}[w_\beta - w_\alpha, u_{\alpha\beta}] + \frac{1}{2}[w_\alpha, w_\beta]) \\
&= : \exp(u'_{\alpha\beta})
\end{aligned}$$

Then $v'_2 = v_2(D^{2'}) = (u', \frac{1}{2}(u')^2)$ is cohomologous to v_2 because

$$v'_2 - v_2 = \partial((w_\alpha), \frac{1}{2}(w_\alpha \times u_{\alpha\beta}) + \frac{1}{2}(w_\alpha \times w_\beta))$$

where $\partial = \delta \pm b$ is the differential of the Čech bicomplex of $\check{C}(J_2)$. Conversely,

supposing $v_2 = (u, \frac{1}{2}u^2)$, $v'_2 = (u', \frac{1}{2}u'^2)$ are cohomologous,

$$v'_2 - v_2 = \partial((w_\alpha), (t_{\alpha\beta})) .$$

Now as $H^0(T) = 0$, $\delta(t) = \frac{1}{2}(u')^2 - \frac{1}{2}u^2$ determines (t) up to adding a Čech coboundary $s_\alpha - s_\beta$ and, using $b\delta = \pm\delta b$ this may be absorbed into (w_α) . Thus we may assume

$$t_{\alpha\beta} = \frac{1}{2}w_\alpha \times u_{\alpha\beta} + \frac{1}{2}w_\alpha \times w_\beta,$$

so that $(D_{\alpha\beta}^2 = \exp(u_{\alpha\beta}))$ and $(D_{\alpha\beta}^{2'} = \exp(u'_{\alpha\beta}))$ are cohomologous as above. This finally completes the proof of Theorem 0.1 for $n=2$.

4. n -TH ORDER

We now complete the proof of Theorem 0.1 in the general n -th order case, following in part the pattern of the case $n=2$ and using induction. However the argument becomes a bit more involved and less direct. Fix an artin local \mathbb{C} -algebra (R_n, m_n)

of exponent n , with reduction (R_{n-1}, m_{n-1}) , etc, and an acyclic open cover (U_α) .

The main point is to associate a morphic hypercocycle

$$v_n = v_n(\mathcal{O}_n/R_n) \in m_n \otimes \check{Z}^0(J_n) ,$$

hence a Kodaira-Spencer homomorphism $\alpha_n(O_n/R_n)$ etc- to an R_n -flat deformation

$\mathcal{O}_n = \mathcal{O}_{X_n}$ of \mathcal{O} . As before we seek v_n of the form

$$v_n = \epsilon(u_n) := (u_n, \frac{1}{2}(u_n)^2, \dots, \frac{1}{n!}(u_n)^n)$$

for some cochain $u_n = (u_{n\alpha\beta}) \in \check{C}^1(T) \otimes m_n$ which is a lift of $u_{n-1} \in \check{C}^1(T) \otimes m_n$

analogously defining v_{n-1} . To this end we start with isomorphisms of algebras

$$\psi_\alpha^n : \mathcal{O}_n(U_\alpha) \xrightarrow{\sim} \mathcal{O}(U_\alpha) \otimes R_n$$

which yield a gluing cocycle by

$$D_{\alpha\beta}^n = \psi_\beta^n (\psi_\alpha^n)^{-1} \in \text{Aut}_{R_n}(\mathcal{O}(U_\alpha \cap U_\beta) \otimes R_n), \quad (4.1)$$

which as above we express in the form

$$D_{\alpha\beta}^n = \exp(t_{n\alpha\beta}), \quad (4.2)$$

This can be done because, assuming inductively that (4.2) holds for $n-1$ and

letting t'_n be an arbitrary lift of t_{n-1} and $t_n = t'_n + \eta_n, \eta_n \in \check{C}^1(T) \otimes m_n^n$, (4.2) can

be rewritten as

$$D_{\alpha\beta}^n = \exp(t'_{n\alpha\beta}) + \eta_n.$$

which can clearly be uniquely solved for η_n .

Now before proceeding with the definition of $u = u_n$ we will consider a Dolbeault analogue, both for its own interest and as motivation for the Čech construction to follow. Consider the DGLA sheaf

$$g^\cdot = (\mathcal{A}^{0,\cdot}(T), \bar{\partial}, [\cdot, \cdot])$$

($\Gamma(g^\cdot)$ is sometimes called the Frolicher-Nijenhuis algebra); as g^\cdot is a soft resolution of T , $J_n(g^\cdot)$ is a soft resolution of $J_n(T)$ which may be used to compute $\mathbb{H}^0(J_n(T))$. As g^0 is soft it is easy to see that, up to shrinking our cover (U_α) we may assume

$$D_{\alpha\beta}^n = \exp(s_\alpha)\exp(-s_\beta)$$

$$s_\alpha \in g^0(U_\alpha) \otimes m_n.$$

Put another way, we may view ψ_α^n above as a holomorphic local trivialisation

$$U_\alpha^n \simeq U_\alpha \times \text{Spec}(R_n)$$

U_α^n = open subset of X_n corresponding to U_α ; on the other hand there is a global ‘ C^∞ trivialisation’ $C : X_n \rightarrow X \times \text{Spec}(R_n)$, and we may set

$$\exp(s_\alpha) = C \circ (\psi_\alpha^n)^{-1} \quad (4.3)$$

Now note that $\bar{\partial}$ extends formally as a derivation on the universal enveloping algebra $U(g^\cdot)$ and we set

$$\phi_\alpha = \exp(-s_\alpha)\bar{\partial}\exp(s_\alpha) = D(ad(s_\alpha))(\bar{\partial}s_\alpha) \quad (4.4)$$

where D is the function

$$D(x) = \frac{\exp(x) - 1}{x} = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}.$$

Note that

$$0 = \bar{\partial}D_{\alpha\beta}^n = \bar{\partial}\exp(s_\alpha)\exp(-s_\beta) + \exp(s_\alpha)\bar{\partial}\exp(\exp(-s_\beta)),$$

hence

$$\exp(-s_\alpha)\bar{\partial}\exp(s_\alpha) = -\bar{\partial}\exp(-s_\beta)\exp(s_\beta);$$

since moreover $\bar{\partial}(\exp(-s_\beta)\exp(s_\beta)) = 0$ we have similarly

$$-\bar{\partial}\exp(-s_\beta)\exp(s_\beta) = \exp(-s_\beta)\bar{\partial}\exp(s_\beta), \quad (4.5)$$

which means precisely that the ϕ_α glue together to a global section

$$\phi \in \Gamma(g^1) = A^{0,1}(T) \otimes m_n.$$

Next, note using (4.4) that

$$\bar{\partial}\phi_\alpha = \bar{\partial}\exp(-s_\alpha)\bar{\partial}\exp(s_\alpha) = \bar{\partial}\exp(-s_\alpha)\exp(s_\alpha)\exp(-s_\alpha)\bar{\partial}\exp(s_\alpha) = -\phi_\alpha\phi_\alpha;$$

recalling that for odd-degree elements $\phi, \psi \in g$, $[\phi, \psi] = \phi.\psi + \psi.\phi$, we conclude

that the integrability equation

$$\bar{\partial}\phi = \frac{-1}{2}[\phi, \phi] \quad (4.6)$$

is satisfied, and consequently

$$\epsilon(\phi) = (\phi, \frac{1}{2}\phi \times \phi, \dots, \frac{1}{n!}\phi \times \dots \times \phi) \in \Gamma(J_n(g)) \otimes m_n$$

is a hypercocycle, which may be used to define a Dolbeault analogue of v_n (automatically morphic, due to the 'exponential' nature of ϵ).

By way of interpretation, note that, as operators,

$$\bar{\partial}(\exp(s_\alpha)) = [\bar{\partial}, \exp(s_\alpha)],$$

therefore clearly

$$\phi_{\cdot} = \exp(-s_{\cdot})\bar{\partial}\exp(s_{\cdot}) - \bar{\partial}. \quad (4.6)$$

What (4.6) means is this: recall the map C above which yields a C^{∞} trivialisation of the deformation X_n/R_n and in particular bundle isomorphisms

$$\mathcal{A}^{0,\cdot}(X) \otimes R_n \simeq \mathcal{A}^{0,\cdot}(X_n/R_n)$$

under which the canonical Dolbeault operator $\bar{\partial}_n$ on the RHS corresponds on the LHS precisely to $\bar{\partial}_0 \otimes 1 + \phi_{\cdot}$. The integrability equation (4.5) reads, on the operator level

$$\bar{\partial}\phi + \phi\bar{\partial} = \phi\phi, \bar{\partial} := \bar{\partial}_0 \otimes 1,$$

i.e. is equivalent to $\bar{\partial}_n^2 = 0$.

Given this, it is now clear how to go backwards. Given $\phi \in A^{0,1}(T) \otimes m_n$ we may define an operator d_n on $\tilde{\mathcal{A}}_n^{0,\cdot} := \mathcal{A}^{0,\cdot}(X) \otimes R_n$ by

$$d_n = \bar{\partial} + \phi_{\cdot},$$

and the integrability equation (4.5) guarantees that $(\tilde{\mathcal{A}}_n^{0,\cdot}, d_n)$ is a complex; by semicontinuity, this complex is clearly exact in positive degrees (because $\tilde{\mathcal{A}}_n^{0,\cdot} \otimes \mathbb{C}$ is) and we may define

$$\mathcal{O}_n = \ker(d_n, \tilde{\mathcal{A}}_n^{0,0}).$$

As d_n is an R_n -linear derivation, \mathcal{O}_n is a sheaf of R_n -algebras. That \mathcal{O}_n is R_n -flat is a consequence of the following easy observation.

Lemma 4.1. *Let R be an artin local ring with residue field k , M an R -module and $M \rightarrow N_{\cdot}$ a flat resolution such that $M \otimes k \rightarrow N_{\cdot} \otimes k$ is also a resolution. Then M is flat.*

Proof. Our assumption implies that $Tor_i(M, k) = 0, i > 0$. Now if P is any finite R -module then P admits a composition series with factors isomorphic to k , hence $Tor_i(M, P) = 0, i > 0$. Finally any R -module Q is a direct limit of its finite submodules and Tor commutes with direct limits, hence $Tor_i(M, Q) = 0$, so M is flat. \square

While the above is sufficient for a Dolbeault proof of Theorem 0.1, it seems desirable to have a translation into the Čech language. To this end, we replace g by the Čech complex $C^\bullet(T)$ which, together with the Čech differential δ and the natural bracket $[\cdot, \cdot]$ forms a DGLA. By analogy with ϕ , we set

$$u. = u_n = \delta(s.) = \exp(-s.)\delta(\exp(s.)) = -\delta(\exp(-s.))\exp(s.).$$

As C is globally defined (albeit nonholomorphic), it commutes with δ hence by (4.3)

$$u_{n\alpha} = (\psi_\alpha^n)^{-1}\delta\psi_\alpha^n$$

so actually $u_n \in \check{C}^1(T)$, i.e. $\bar{\partial}(u_n) = 0$. In particular,

$$\bar{\partial}(\exp(-s.)\delta(\exp(s.))) = -\exp(-s.)\bar{\partial}\delta(\exp(s.)).$$

On the other hand $\delta(\phi.) = 0$ yields

$$\delta(\exp(-s.))\bar{\partial}(\exp(s.)) = -\exp((-s.))\delta\bar{\partial}(\exp(s.)).$$

As δ and $\bar{\partial}$ commute it follows that

$$\delta(\exp(-s.))\bar{\partial}((\exp(s.)) = \bar{\partial}((\exp(-s.))\delta(\exp(s.)).$$

Hence

$$u.\phi. = \exp(-s.)\delta(\exp(s.))\exp(-s.)\bar{\partial}(\exp(s.)) = -\delta(\exp(-s.))\bar{\partial}(\exp(s.))$$

$$= -\bar{\partial}(\exp(-s.))\delta(\exp(s.)) = \phi.u.,$$

i.e. $u., \phi.$ commute:

$$[u., \phi.] = 0 \quad (4.8)$$

Now as above we have formally that

$$\delta(u.) = \delta(\exp(-s.))\delta(\exp(s.)) = \frac{-1}{2}[u., u.],$$

and therefore $v_n = \epsilon(u_n) \in \check{C}^0(J_n(T))$ is a morphic hypercocycle, which may be used to define the required Kodaira-Spencer homomorphism $\alpha_n(X_n/R_n) : R_n^u \rightarrow R_n$.

The interpretation of u_n is analogous to that of ϕ : i.e. the operator

$$\delta + u_n : \check{C}^\bullet(\mathcal{O}) \otimes R_n \rightarrow \check{C}^{\bullet+1}(\mathcal{O}) \otimes R_n$$

corresponds to the coboundary operator on $\check{C}^\bullet(\mathcal{O}_n)$ under the local trivialisation (ψ_α^n) above. Thus to reverse this construction we may proceed analogously as in the Dolbeault case. Firstly we represent a morphic element $v_n \in \mathbb{H}^0(J_n) \otimes m_n$ in the form

$$v_n = \epsilon(u_n), u_n \in \check{C}^1(T) \otimes m_n \quad (4.9)$$

where $u. = u_n$ satisfies the Čech integrability equation

$$\delta(u.) = \frac{-1}{2}[u., u.]. \quad (4.10)$$

Now thanks to (4.9), the deformed coboundary operator

$$\delta' = \delta + u_n : \check{C}^\bullet(\mathcal{O}) \otimes R_n \rightarrow \check{C}^{\bullet+1}(\mathcal{O}) \otimes R_n$$

satisfies $(\delta')^2 = 0$, thus making $(\check{C}^\bullet(\mathcal{O}) \otimes R_n, \delta')$ as well as its sheafy version $(\check{C}^\bullet(\mathcal{O}) \otimes R_n, \delta')$ into complexes where the latter is exact in positive degrees. Hence as before,

$$\mathcal{O}_n = \ker(\delta', \check{C}^0(\mathcal{O}) \otimes R_n)$$

is a sheaf of flat R_n -algebras yielding a flat deformation

$$\Phi_n(v_n) = X_n/R_n = \text{Specan}(\mathcal{O}_n).$$

It is worth noting that the Čech construction yields the same deformation as the Dolbeault one: this follows easily from (4.8). Also, it is clear from the construction that either ϕ or u_n determines the deformation X_n/R_n up to isomorphism.

Now we may easily complete the proof as in Sect.3. First, taking the element v_n corresponding to the identity on $\mathbb{H}^0(J_n)$, we obtain a corresponding deformation X_n^u/R_n^u . Next, given any X_n/R_n , with corresponding u_n, α_n , it is clear that

$$\alpha_n = \alpha_n^*(X_n^u/R_n^u).$$

Since a deformation is determined by its α_n it follows that

$$X_n/R_n \simeq \alpha_n^*(X_n^u/R_n^u).$$

Thus X_n^u/R_n^u is n -universal. Finally it is clear by construction that for different n these are mutually compatible so the limit \hat{X}_n^u/\hat{R}_n^u exists and is formally universal, completing the proof of Theorem 0.1.

Remark See [R3] for an 'interpretation' of the construction of X_n^u .

5. GENERALIZATIONS

Let g be a sheaf of \mathbb{C} - Lie algebras on X with $H^0(g) = 0$, E a g -module. Replacing g by its unique quotient acting faithfully on E , we may assume E is faithful. We also assume X, g, E are reasonably tame so cohomology can be computed by Čech complexes. We further assume g and E admit compatible soft resolutions g^\cdot, E^\cdot where g^\cdot is a DGLA acting on E^\cdot . Typically, E will have some additional structure and g will coincide with the full Lie algebra of infinitesimal automorphisms of

the structure: e.g. when E is a ring g may be the algebra of internal derivations. For any artin local \mathbb{C} -algebra (R, m) , we have a Lie group sheaf $Aut_R^\circ(E \otimes R)$ of R -linear automorphisms of $E \otimes R$ which act as the identity on $E = (E \otimes R) \otimes_R \mathbb{C}$, and we assume given a Lie subgroup sheaf

$$G_R \subset Aut_R^\circ(E \otimes R)$$

with Lie algebra $g \otimes m$, which coincides - by definition if you will - with the subgroup of structure-preserving automorphisms in $Aut_R^\circ(E \otimes R)$. Then the above constructions, being essentially formal in nature, carry over to this setting essentially verbatim, yielding n -universal deformations E_n^u/R_n^u , $n \geq 1$, and a formally universal deformation \hat{E}^u/\hat{R}^u .

Examples (cf. [R5])

5.a. E is a simple locally free finite-rank \mathcal{O}_X -module and g is the algebra of all traceless \mathcal{O}_X -linear endomorphisms of E . The deformation obtained is the usual universal deformation of E as \mathcal{O}_X -module.

Subexamples

5.a₁. \mathcal{O}_X is the ring of locally constant functions on the topological space X assumed 'nice', e.g. a manifold. In this case E is a local system (i.e. a π_1 representation), and we obtain its universal deformation as such.

5.a₂. \mathcal{O}_X is the sheaf of holomorphic functions on a complex manifold (or regular functions on a proper \mathbb{C} -scheme). In this case E is a (holomorphic) vector bundle and we obtain its universal formal deformation as such.

5.b. Let $Y \subset X$ be an embedding of compact complex manifolds, $g = T_{X/Y}$ the algebra of vector fields on X tangent to Y along Y , which may be identified with the algebra of infinitesimal automorphisms(i.e. internal derivations) of \mathcal{O}_X preserving

the subsheaf \mathcal{I}_y . Assuming $H^0(T_{X/Y}) = 0$, we obtain the universal deformation of the pair (X, Y) .

The case of general holomorphic map $f : Y \rightarrow X$ may be treated in a similar way using the algebra T_f (cf. [R2]); in fact it is sufficient for many purposes to replace f by the embedding of its graph in $Y \times X$ (cf. [R4]). On the other hand the case of deformations of $Y \rightarrow X$ with X fixed requires the DGLA formalism and will be taken up in [R3].

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